Multiple Choice

1) (A) The constant term is 10, so, test $P(\pm 2)$ only.
$P(2)=8-20+22-10=0, x-2$ is a factor

2) (B) $\log_a 8=1.893=3\log_a 2, \therefore \log_a 4=2\log_a 2$
$=\frac{2}{3}\times 1.893=1.26$

3) (B) $\angle DAB=70^\circ$ (opposite angles in a cyclic quad)
$\angle TAB=65^\circ$ (alternate segment angles)
$\angle TAD=\angle TAB+\angle DAB=65^\circ+70^\circ=135^\circ$

4) (C) $\frac{1}{2}$

5) (D) Only (D) has horizontal asymptote $y=-2$

6) (D) $y'=\frac{2x}{4}, \therefore m_1=\frac{1}{p}, m_2=-p.$
Eqn $y-\frac{1}{p^2}=-p\left(x-\frac{2}{p}\right)=-px+2$
$p^2y+p^3x=2p^2+1$

7) (A) $-1\le\frac{3}{x}\le 1, \therefore x\ge 3$ or $x\le -3$

8) (C), $A=\pi r^2, \frac{dA}{dt}=\frac{dA}{dr}\frac{dr}{dt}=2\pi r\times 5=150\pi$

9) (C) For $\left(x^n+\frac{1}{x^{n+1}}\right)^7$, the constant term occurs
when $(x^n)^{7-k}\left(\frac{1}{x^{n+1}}\right)^k=x^0$.
$\therefore 7n-kn-(n+1)k=0$
$\therefore k=\frac{7n}{2n+1}$
As k is an integer, $n=3$ (and the constant is ${}^7C_3$)

10) (B) You can pick any 3 squares by $9C3=84$ ways.
As there are only 8 ways to end up in the same row, column or diagonal, $\Pr=\frac{8}{84}=\frac{2}{21}$

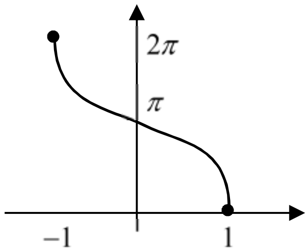
Question 11

(a) $x=\frac{3(-4)+2(1)}{2+3}=-2$

(b) $\frac{d}{dx}\tan^{-1}(x^3)=\frac{3x^2}{1+x^6}$

(c) $\frac{2x}{x+1}>1$
$2x(x+1)>(x+1)^2$
$(x+1)(2x-(x+1))>0$
$(x+1)(x-1)>0$
$\therefore x<-1$ or $x>1$

(d)



(e) Let $x=u^2-1, dx=2udu.$
When $x=0, u=1$; when $x=3, u=2$
$$\int_0^3 \frac{x}{\sqrt{x+1}}dx=\int_1^2\frac{u^2-1}{u}2udu=2\left[\frac{u^3}{3}-u\right]_1^2=\frac{8}{3}$$

(f) $\int \sin^2 x\cos x dx=\frac{\sin^3 x}{3}+C$

(g) (i) ${}^8C_3\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^5$

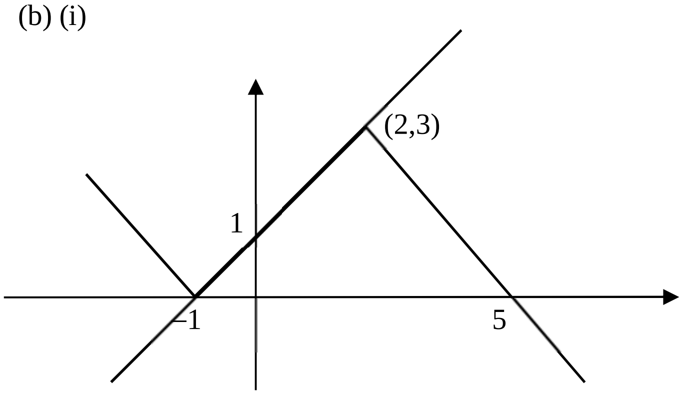
(ii) $\left(\frac{4}{5}\right)^8$

(iii) $1-\left(\frac{4}{5}\right)^8$

Question 12

(a) $\angle ABC = \frac{1}{2}(360° - 100°) = 80°$ (angle at the circumference is half the angle at the centre subtending the same arc)

(b) (i)



(ii) $-1 \le x \le 2$

(c) (i) Given $\pi\int_{-1}^{h}(1-x^2)dx = 2\pi\int_{h}^{1}(1-x^2)dx.$

$$\left[x-\frac{x^3}{3}\right]_{-1}^{h} = 2\left[x-\frac{x^3}{3}\right]_{h}^{1}$$

$$h-\frac{h^3}{3}+\frac{2}{3} = 2\left(\frac{2}{3}-h+\frac{h^3}{3}\right)$$

$$3h-h^3+2 = 4-6h+2h^3$$

$$3h^3-9h+2=0$$

(ii) Let $f(h)=3h^3-9h+2$

$$f'(h)=9h^2-9$$

$$x_1 = 0-\frac{f(0)}{f'(0)} = -\frac{2}{-9} = \frac{2}{9}$$

(d) $t = 4-e^{-2x}$

$$e^{-2x} = 4-t$$

$$-2x = \ln(4-t)$$

$$x = -\frac{1}{2}\ln(4-t)$$

$$\frac{dx}{dt} = \frac{1}{2(4-t)}$$

$$\frac{d^2x}{dt^2} = \frac{1}{2(4-t)^2} = \frac{1}{2e^{-4x}} = \frac{e^{4x}}{2}$$

(e) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2$

Question 13

(a) Using $v^2 = n^2(A^2-x^2)$,

$16 = n^2(A^2-4)$ (1)

$9 = n^2(A^2-25)$ (2)

$(1)-(2)$ gives $7 = 21n^2, \therefore n = \frac{1}{\sqrt{3}}$.

$$\text{Period} = \frac{2\pi}{n} = 2\sqrt{3}\pi$$

(b) (i) $(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n$ (3)

$$(1-x)^n = \binom{n}{0}-\binom{n}{1}x+\binom{n}{2}x^2-\ldots+(-1)^n\binom{n}{n}x^n$$

$=\binom{n}{0}-\binom{n}{1}x+\binom{n}{2}x^2-\ldots+\binom{n}{n}x^n$, if n is even (4)

$(3)+(4)$ gives

$$(1+x)^n+(1-x)^n = 2\binom{n}{0}+2\binom{n}{2}x^2+\ldots+2\binom{n}{n}x^n.$$

(ii) Differentiate wrt x,

$$n(1+x)^{n-1}-n(1-x)^{n-1} = 2\times 2\binom{n}{2}x+4\times 2\binom{n}{4}x^3+\ldots$$

$$+n\times 2\binom{n}{n}x^{n-1}$$

$$n\left[(1+x)^{n-1}-(1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x+4\binom{n}{4}x^3+\ldots\right.$$

$$\left.+n\binom{n}{n}x^{n-1}\right]$$

(iii) Let $x=1$,

$$n2^{n-1} = 2\left[2\binom{n}{2}+4\binom{n}{4}+\ldots+n\binom{n}{n}\right]$$

$$n2^{n-1} = 4\left[\binom{n}{2}+2\binom{n}{4}+\ldots+\frac{n}{2}\binom{n}{n}\right]$$

$$n2^{n-3} = \binom{n}{2}+2\binom{n}{4}+\ldots+\frac{n}{2}\binom{n}{n}$$

(c) (i) Let $y=0, t=\frac{2V\sin\theta}{g}$.

Sub to x gives $x = \frac{2V^2\sin\theta\cos\theta}{g} = \frac{V^2\sin 2\theta}{g}$

(ii) If $V^2 < 100g$ then $x < 100\sin 2\theta$.

Since $\sin 2\theta < 1, x < 100$.

(iii) If $V^2 = 200g$ and $x \ge 100$

$200\sin 2\theta \ge 100$

$\sin 2\theta \ge \frac{1}{2}$

From the graph of $\sin x, \sin x \ge \frac{1}{2}$ for $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$

$\therefore \frac{\pi}{6} \le 2\theta \le \frac{5\pi}{6}, \therefore \frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$.

(iv) The greatest height is reached when $\dot{y} = 0$,

$V\sin\theta - gt = 0$

$t = \frac{V\sin\theta}{g}$

$\therefore \text{Max } y = \frac{V^2\sin^2\theta}{g} - \frac{1}{2}g\frac{V^2\sin^2\theta}{g^2} = \frac{V^2\sin^2\theta}{2g}$

$= \frac{200g\sin^2\theta}{2g} = 100\sin^2\theta = 100\sin^2\frac{5\pi}{12}$, since

for $\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$, $\sin\frac{5\pi}{12}$ is largest.

Question 14

(a) Let $n = 1, 8^3 + 6 = 518 = 74 \times 7, \therefore$ divisible by 7

Assume $8^{2n+1} + 6^{2n-1} = 7M$, where M is an integer,

i.e. $8^{2n+1} = 7M - 6^{2n-1}$.

Required to prove that $8^{2n+3} + 6^{2n+1}$ is divisible by 7.

$8^{2n+3} + 6^{2n+1} = 64 \times 8^{2n+1} + 36 \times 6^{2n-1}$

$= 64\left(7M - 6^{2n-1}\right) + 36 \times 6^{2n-1}$

$= 64 \times 7M - 28 \times 6^{2n-1}$

$= 7\left(64M - 4 \times 6^{2n-1}\right)$, which is divisible by 7.

$\therefore$ By the principle of Induction, it is true that $8^{2n+1} + 6^{2n-1}$ is divisible by 7.

(b) (i) For the tangent at P, $m = \frac{dy/dp}{dx/dp} = p$

$y - p^2 = p(x - 2p), \therefore y = px - p^2$

Sub to $x^2 = -4ay$ gives $x^2 = -4a\left(px - p^2\right)$

$\therefore x^2 + 4apx - 4ap^2 = 0$.

(ii) $M \left| \begin{array}{l} x = \frac{\sum\alpha}{2} = \frac{-4ap}{2} = -2ap \\ y = p(-2ap) - p^2 = -p^2(2a+1) \end{array}\right.$

(iii) Sub to $x^2 = -4y$ gives $4a^2p^2 = 4p^2(2a+1)$

$\therefore a^2 - 2a - 1 = 0, \therefore a = 1 \pm \sqrt{2}$

(c) (i) $\frac{d}{dt}F(t)e^{0.4t} = F'(t)e^{0.4t} + 0.4F(t)e^{0.4t}$

$= \left(50e^{-0.5t} - 0.4F(t)\right)e^{0.4t} + 0.4F(t)e^{0.4t}$

$= 50e^{-0.1t}$

(ii) $F(t)e^{0.4t} = \int 50e^{-0.1t}dt = -500e^{-0.1t} + C$

When $t = 0, F(0) = 0, \therefore C = 500$

$\therefore F(t)e^{0.4t} = 500\left(1 - e^{-0.1t}\right)$

$\therefore F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$

(iii) $F'(t) = 0$ gives $50e^{-0.5t} = 0.4F(t)$

$\therefore 50e^{-0.5t} = 0.4 \times 500\left(e^{-0.4t} - e^{-0.5t}\right)$

$e^{-0.5t} = 4\left(e^{-0.4t} - e^{-0.5t}\right)$

$4e^{-0.4t} = 5e^{-0.5t}, \therefore e^{0.1t} = \frac{5}{4}$

$t = 10\ln\frac{5}{4}$